

Relative Stability of Numerical Methods

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Abstract: Through the use of numerical methods many problems can be solved that would otherwise be thought to be insoluble and the results of ODE possessing initial value problems can be found easily. Analytically some valuable numerical methods like Euler's method is analyzed while Runge Kutta method presented here gives a better exactness. Derivatives of Higher order are not essential for tabulation. Advantages and disadvantages of these methods are presented in this paper and thus comparing solutions obtained by them. These descriptions provide an introduction to numerical methods for the solution of physical problems.

Key words: Euler's Method, Runge Kutta method, ODE, IVP, step size.

1 Introduction

The method which varies can be explained through Differential equations. The characteristics of the differential equations have been implicit and considered by many engineers and many difficult troubles can moderately be explained more perfectly with the help of mathematical terms. The general solution of nth order ODE possesses n constants. To have particular solution, the situations are explained at one, at two or more points and accordingly decide the nature of ODE as initial value and boundary value problems. In this particular paper, Euler's method, modified Euler's method and Runge Kutta methods have been discussed as initial value problems respectively. [8], [4], [6], [5], and [1]

2 Euler's Method

To solve a first order ordinary differential equation having IVP, a basic and entirely numerical method Euler's method is used.

Considering the first order ordinary differential equation

$$\frac{dy}{dx} = \phi(x, y) \dots\dots\dots(1)$$

Let the solution of (1) be

$$y = F(x) \dots\dots\dots(2)$$

Taking all the points $x_0, x_1, x_2, \dots, x_n$ equidistant with the same interval h .

Let different points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ be on the curve (2) and If (x_{n+1}, y_{n+1}) lies on (2) then $y_{n+1} = F(x_{n+1})$

As $x_{n+1} = x_n + h$ therefore $y_{n+1} = F(x_n + h)$

Using Taylor's series

$$y_{n+1} = F(x_n) + h F'(x_n) + \frac{1}{2} h^2 F''(x_n) + \dots(3)$$

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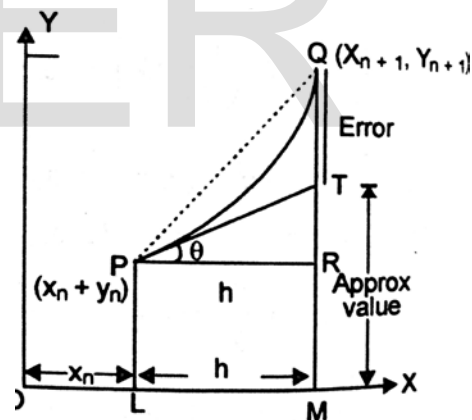
As h is very small

$$y_{n+1} = F(x_n) + h F'(x_n)$$

$$y_{n+1} = F(x_n) + h f(x_n, y_n) \dots\dots\dots(4)$$

This formula (4) can be used to find y_{n+1} where y_n is known.

Equation (4) is called Euler's method.



2.1 Drawbacks in Euler's Method

The input information is rarely exact since it comes from some measurement or the other and the method also introduces further error [3] and [7]. Rounding errors arise from the process of rounding off the numbers during the computation and Truncation errors are caused by using approximate results or on replacing an infinite process by a finite one. Since we have neglected $\frac{1}{2} h^2 F''(x_n)$ and higher powers of h from formula (3) there will be a larger truncation error in y_{n+1} . Therefore Euler's method is not used in practical problems.

2.2 Geometrical Meaning

Let PQ represents the curve of solution $y = F(x)$. The ordinate of P i.e. y_n is known. Then

Ordinate Q i.e. $y_{n+1} = MQ = MR + RQ = PL + RT + TQ$ (TQ Error)

$$= y_n + h(\tan x_n)$$

$$= y_n + h\left(\frac{dy}{dx}\right) = y_n + h f(x_n, y_n)$$

3 Modified Euler's Method

In the Euler's method, the curve of the solution in the interval PQ is approximated by the tangent at P such that at Q, we have

$$y_1 = y_0 + hf(x_0, y_0)$$

Then the slope of the curve of the solution through Q is computed and a tangent at Q is drawn meeting the ordinate through $x_0 + 2h$ in $(x_0 + 2h, y_2)$.

Now we find a better approximation $y_1^{(1)}$ of $y(x_0 + h)$ by taking the slope of the curve as the mean of the slopes of the tangents at P and Q, i.e.,

$$y_1^{(1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_0 + h, y_1)] \dots \dots \dots (5)$$

As y_1 is not known so we take y_1 as found in usual Euler's method and insert it in (5) to obtain the first modified value $y_1^{(1)}$.

Therefore modified Euler's method is given by

$$y_1 = y_0 + \frac{h}{2} (k_1 + k_2)$$

where $k_1 = hf(x_0, y_0)$ and $k_2 = hf(x_0 + h, y_0 + k_1)$

Similarly we proceed to calculate y_2 and y_3 and so on.

This is the modified Euler's method which gives great improvement in accuracy over the original method.

4 Runga Kutta Method

The Taylor's series method of solving differential equations numerically is restricted by the labour involved in finding the higher order derivatives. However there is a class of methods known as Runga Kutta methods which do not require the calculations of higher order derivatives and give greater accuracy. The Runga-Kutta formulae possess the advantage of requiring only the function values at some selected points. These methods agree with Taylor's series solution upto the term in h^r where r differs from method to method and is called the order of that method.

The following are the orders of Runge Kutta Method as listed below:

- ❖ Runge Kutta method of order one is called Euler's method.

- ❖ Runge Kutta method of order two is the same as modified Euler's Method.
- ❖ The fourth order Runge Kutta method is called classical Runge Kutta method.

In this paper, we shall only consider the fourth order Runge Kutta method.

4.1 Working rule of the Fourth Order Runge Kutta Method

This method is most commonly used and many numerical analyst rely on this method because it is quite stable, accurate and easy to program.

To find the increment k of y relative to increment h of x by Runge Kutta Method from

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \text{ is as follows:}$$

Calculate successively $k_1 = hf(x_0, y_0), k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1)$

And $k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2), k_4 = hf(x_0 + h, y_0 + k_3)$

And finally compute

$$k = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

which gives the required approximate value as $y_1 = y_0 + k$

5 Automatic Error management

In the case of Uniform grid spacing, the moments of time steps are superior sometimes. But often we deal with troubles where there is a sense of changeable grid size. But the problem is to know when to change the stepsize? If we take up a plan of order r and $r+1$ have, we can take the difference between these two to be the error in the plan, and thus we can take the step size smaller if we prefer a smaller error, or larger if we can bear a larger error. But For Automatic error management it's "useless" to compute $r+1$ th order as with the help of Runge Kutta method we can take a fifth order method and a fourth order method, using the same k 's. only.

6 Numerical Experiments

To regulate and confirm the accuracy of the methods for resolution in ordinary differential equations having initial value problems, methods were compared. The number of iterations decides the correctness whereas the size of interval decides the accuracy

Example 1:

The problem discussed here is $F(x, y) = x^2 + y$, the linear first order initial value problem, with equal interval. The results obtained shown in Table 1, the comparison of the methods to the exact solution and the error incurred respectively.

7 Table of Results

Table 1: The comparative Result Analysis of Runga Kutta Method and Euler's:

8 Conclusion

The Euler method does not take into account the curvature of the solution, so it tends to give different results depending on the step size. RK, depending on the order, takes into account the curvature. And this makes the estimated "next step" more accurate. Consequently, one can examine that this particular method gives a better approximate consequence Basically, if you are pretending a straight line is a good approximation of a curve (Euler) you will always be overshooting your solution. But if you take account of the curvature (RK) you can follow the curve. On Comparison it can be concluded that Euler sees where the problem is and Runge-Kutta sees where the problem is going to be.

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Sr no	Step size	By Euler's method	By Runga Kutta Method	error incurred
1	0.1	1.1	1.1105	0.0105
2	0.2	1.21	1.2432	0.0332
3	0.3	1.351	1.4004	0.0494
4	0.4	1.5161	1.5846	0.0685
5	0.5	1.7077	1.7988	0.0911
6	0.6	1.9385	2.0460	0.1075
7	0.7	2.2023	2.3298	0.1275
8	0.8	2.5025	2.6534	0.1509
9	0.9	2.8428	3.0227	0.1799
10	1.0	3.2271	3.4409	0.2138

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